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A necessary and sufficient condition for best proximity point in complete metric spaces with applications to analytic functions

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Abstract

This paper proposes a best proximity point theorem for non-self mappings in complete metric spaces, providing necessary and sufficient conditions for their existence. A fixed point theorem is introduced as a corollary and these results are applied to analytic functions of a complex variable, showcasing their relevance to broader mathematical and applied contexts.

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1. Introduction

Fixed point theory is one of the fundamental areas of mathematics with wide-ranging applications in analysis, topology, optimization, dynamical systems, etc. A fixed point of a self-mapping T defined on a set X is a point $x \in X$ such that $T(x) = x$. The study of fixed points has profound implications in solving equations, analyzing iterative methods, stability analysis, etc. Metric fixed point theory, a significant

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branch of fixed point theory, focuses on the existence and properties of fixed points in metric spaces. The following theorem, commonly referred as Banach contraction principle, was established in 1922, a cornerstone of metric fixed point theory. This result states that:

Theorem 1.1 ([3]). *Let (X, d) be a complete metric space and let T be a contraction map on X, such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq r \cdot d(x, y)$$

where $r \in [0, 1)$. Then the self-map T has a unique fixed point x^*. Furthermore, for any initial point $x_0 \in X$, the iterative sequence $<x_n>$ defined by $x_{n+1} = T(x_n)$ converges to x^*.

There are numerous intriguing variations and extensions of Banach contraction principle in the literature; refer to [26, 9, 23, 14, 30, 25, 8, 33, 34, 20, 24, 19]. Most classical results in metric fixed point theory including Edelstein Fixed Point Theorem [16] (1961), Kannan Fixed Point Theorem [21] (1969), Chatterjea Fixed Point Theorem [13] (1972), Caristi Fixed Point Theorem [10] (1976) etc, and its numerous generalizations [27, 28, 31, 32, 11, 12, 29], deal with self-mapping. However, when considering non-self mapping $T : A \to B$, it is not guaranteed that there will always exist a fixed point.
In the absence of fixed points, attention shifts to finding an element $x \in A$ that is, in some sense, closest to Tx. This motivates the study of best approximation and best proximity point theorems. While best approximation theorems establish the existence of approximate solutions, they do not always ensure optimality. In contrast, best proximity point theorems provide sufficient conditions for the existence of approximate solutions that are also optimal, offering a more reliable framework for non-self mappings.
For a non-self mapping $T : A \to B$, the best proximity point is an optimal approximate solution of the equation $Tx = x$ which satisfies the condition $d(x, Tx) = d(A, B)$, where $d(A, B)$ is the distance between the sets A and B. Best proximity point theorems naturally generalize fixed point theorems, as a best proximity point becomes a fixed point when the mapping in question is a self-mapping. These theorems play a pivotal role in extending the application of fixed point theory to a broader class of problems. In a scenario where exact solutions are unattainable, best proximity points offer optimal approximate solutions. For example, in control systems, best proximity points are instrumental in designing controllers that bring the system's state as close as possible to the desired state when achieving exact control is not feasible. Beyond control theory, best proximity points have numerous direct and indirect applications across diverse fields, including applied mathematics, game theory, engineering, economics, and computer science, demonstrating their importance in addressing complex, real-world challenges.
Numerous best proximity point theorems for various types of contractions have been studied in references [17, 15, 1, 22, 4, 5, 7, 35, 18, 2, 6]. These theorems provide a framework for ensuring the existence of best proximity points for non-self mappings under specific conditions, along with an iterative process

to approximate these points. However, a key limitation of majority of these existing theorems is that they generally provide these sufficient conditions for the existence of best proximity points instead of necessary conditions. This limitation highlights an important gap in the literature, emphasizing the need for further research to establish conditions that are both necessary as well as sufficient for the existence of best proximity points.

Proposition 1.2. *Let $f : X \to Y$. Then f has a best proximity point if and only if there is a constant map $g : Y \to X$ such that*

$$d(g(f(x)), f(g(f(x)))) = d(X, Y) \qquad \textit{for all } x \in X.$$

Proof. (Sufficiency): By hypothesis there exists $u \in X$ and $g : Y \to X$ such that $g(y) = u$ for all $y \in Y$ and so $g(f(x)) = u$ for all $x \in X$.
Therefore $d(g(f(x)), f(g(f(x)))) = d(X, Y)$ for all $x \in X$, which implies $d(u, f(u)) = d(X, Y)$.
Thus u is best proximity point of mapping f.

(Necessity): Suppose that f has a best proximity point, say $u \in X$. Then

$$d(u, f(u)) = d(X, Y). \tag{1.1}$$

Define $g : Y \to X$ as $g(y) = u$ for all $y \in Y$.
Then $g(f(x)) = u$ for all $x \in X$ and so from (1.1) we have

$$d(u, f(u)) = d(X, Y)$$

This implies $d(g(f(x)), f(g(f(x)))) = d(X, Y)$ for all $x \in X$. □

The intent of this paper is to offer a best proximity point theorem in the spirit of the aforementioned proposition 1.2. In Section 2, we present the proof of the best proximity point theorem within the framework of the earlier proposition. Additionally, this section explores various illustrative examples to demonstrate the applicability and validity of the theorem. Furthermore, sufficient conditions are provided to guarantee the uniqueness of the best proximity point. Moreover, a fixed point theorem is introduced, offering a comprehensive characterization through necessary and sufficient conditions for the existence of fixed points. Finally, Section 3 is devoted to an application of the proposed theorem in the context of analytic functions of a complex variable. This application highlights the broader implications and utility of the theorem in mathematical analysis and related fields, showcasing its potential to address complex problems in diverse domains.

2. Main Results

In this section, we prove the best proximity point theorem which extends the concept of best proximity points and provide a fixed point theorem that provides both necessary and sufficient conditions for the existence of fixed points.

Theorem 2.1. *Let (X, d) and (Y, d) be complete metric spaces. Let $f : X \to Y$ be a continuous mapping. Then f has a best proximity point in X if and only if there exists a continuous mapping $g : Y \to X$ which satisfies the following conditions:*

(i) $d(g(f(x)), f(g(f(x)))) = d(X, Y)$ *for all* $x \in X$.

(ii) *For all* $x, y \in X$, $d(g(f(x)), g(f(y))) \leq \alpha d(x, y)$.

(iii) *For all* $x', y' \in Y$, $d(f(g(x')), f(g(y'))) \leq \alpha d(x', y')$.

where $\alpha \in (0, 1)$.

Proof. (Sufficiency): Suppose there is a continuous mapping $g : Y \to X$ which satisfies conditions (i), (ii) and (iii).

Let $x_0 \in X$ be arbitrary such that $f(x_{2n}) = x_{2n+1}$ and $g(x_{2n+1}) = x_{2n+2}$ for all $n \geq 0, n \in \mathbb{Z}$.

Define two sequences $\langle x_{2n} \rangle$ and $\langle x_{2n+1} \rangle$ in X and Y respectively such that

$$x_{2n} = (gf)^n(x_0) \text{ and } x_{2n+1} = f(gf)^n(x_0).$$

Then from (ii) and (iii),

$$d(g(f(x_0)), g(f(x_2))) \leq \alpha d(x_0, x_2)$$

implies that $d(x_2, x_4) \leq \alpha d(x_0, x_2)$. Also

$$\begin{aligned} d(x_4, x_6) &= d(g(f(x_2)), g(f(x_4))) \\ &\leq \alpha d(x_2, x_4) \\ &\leq \alpha^2 d(x_0, x_2). \end{aligned}$$

In general, $d(x_{2n}, x_{2n+2}) \leq \alpha^n d(x_0, x_2)$.

Similarly

$$d(f(g(x_1)), f(g(x_3))) \leq \alpha d(x_1, x_3)$$

implies that $d(x_3, x_5) \leq \alpha d(x_1, x_3)$. Also

$$\begin{aligned} d(x_5, x_7) &= d(f(g(x_3)), f(g(x_5))) \\ &\leq \alpha d(x_3, x_5) \\ &\leq \alpha^2 d(x_1, x_3). \end{aligned}$$

In general, $d(x_{2n+1}, x_{2n+3}) \leq \alpha^n d(x_1, x_3)$.
As $\alpha \in (0,1)$, both the sequences $\langle (gf)^n(x_0) \rangle$ and $\langle f(gf)^n(x_0) \rangle$ are Cauchy sequences in X and Y respectively. The completeness of X and Y implies that both these sequences are convergent. Let $u \in X$ and $v \in Y$ such that $\langle (gf)^n(x_0) \rangle$ converges to u and $\langle f(gf)^n(x_0) \rangle$ converges to v. Since f is a continuous mapping, $\langle f(gf)^n(x_0) \rangle$ converges to $f(u)$ and uniqueness of limit implies $f(u) = v$.
A similar argument asserts that $g(v) = u$.
Also form (i) for $x = u$, we have

$$d(g(f(u)), f(g(f(u)))) = d(X, Y)$$

implies that $d(u, v) = d(X, Y)$, or

$$d(u, f(u)) = d(X, Y).$$

This shows that u is proximity point of mapping f.

(Necessity): Suppose that mapping f has best proximity point, say $u \in X$. Then

$$d(u, f(u)) = d(X, Y). \tag{2.1}$$

Define $g : Y \to X$ as $g(y) = u$ for all $y \in Y$. Then $g(f(x)) = u$ for all $x \in X$, and so from (2.1)

$$d(g(f(x)), f(g(f(x)))) = d(X, Y) \text{ for all } x \in X$$

implies that (i) is satisfied.
For any $\alpha \in (0,1)$, we have for all $x, y \in X$,

$$\begin{aligned} d(g(f(x)), g(f(y))) &= d(u, u) \\ &= 0 \\ &\leq \alpha d(x, y). \end{aligned}$$

Similarly for all $x', y' \in Y$,

$$\begin{aligned} d(f(g(x')), f(g(y'))) &= d(f(u), f(u)) \\ &= 0 \\ &\leq \alpha d(x', y'). \end{aligned}$$

Thus (ii) and (iii) holds. This completes the proof. □

The following example illustrates the preceding Theorem 2.1.

Example 2.2. Consider the Euclidean two space $\mathbb{R}^2$ with the metric d, defined as

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| \text{ for all } (x_1, x_2) \text{ and } (y_1, y_2) \text{ in } \mathbb{R}^2.$$

Let $X = \{(0, 1), (1, 0)\}$ and $Y = \{(0, -1), (-1, 0)\}$. Define $f : X \to Y$ and $g : Y \to X$ as

$$f((x_1, x_2)) = (-1, 0) \text{ for all } (x_1, x_2) \in X,$$

and

$$g((y_1, y_2)) = (0, 1) \text{ for all } (y_1, y_2) \in Y.$$

Then it is very easy to show that all the conditions of the preceding theorem are satisfied and indeed $(0, 1)$ and $(1, 0)$ are best proximity points of the mapping f.

The preceding example demonstrates that while Theorem 2.1 provides necessary and sufficient conditions for the existence of best proximity points of a continuous mapping from a complete metric space to another complete metric space, it does not guarantee their uniqueness. To address the uniqueness of best proximity points, we present the following corollary.

Corollary 2.3. *Let X, Y, f and g be same as in Theorem 2.1. If $u \in X$ and $v \in Y$ such that for all $x \in X$ and for all $y \in Y$*

$$d(u, v) \leq d(x, y) \text{ implies } u = g(v). \tag{2.2}$$

Then f has unique best proximity point.

Proof. Suppose mapping f has two best proximity points, say u_1 and u_2. Then

$$d(u_1, f(u_1)) = d(X, Y),$$

and

$$d(u_2, f(u_2)) = d(X, Y).$$

As for all $x \in X$ and for all $y \in Y$,

$$d(u_1, f(u_1)) \leq d(x, y)$$

(2.2) implies $u_1 = g(f(u_1))$. Similarly, by employing the same procedure, we can show that $u_2 = g(f(u_2))$. Then

$$\begin{aligned} d(u_1, u_2) &= d(g(f(u_1)), g(f(u_2))) \\ &\leq \alpha d(u_1, u_2) \qquad \{\text{using } (ii)\} \end{aligned}$$

which implies $d(u_1, u_2)(1 - \alpha) \leq 0$. As $\alpha \in (0, 1), d(u_1, u_2) = 0$ implies $u_1 = u_2$, which proves the uniqueness of the best proximity point for the mapping f. □

The following example illustrates the preceding result.

Example 2.4. Consider the Euclidean two space $\mathbb{R}^2$ with the metric d, defined as

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| \text{ for all } (x_1, x_2) \text{ and } (y_1, y_2) \text{ in } \mathbb{R}^2.$$

Let $X = \{(0, 1), (2, 0)\}$ and $Y = \{(0, -1), (-2, 0)\}$. Define $f : X \to Y$ and $g : Y \to X$ as

$$f((x_1, x_2)) = (0, -1) \text{ for all } (x_1, x_2) \in X,$$

and

$$g((y_1, y_2)) = (0, 1) \text{ for all } (y_1, y_2) \in Y.$$

Then $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are given by

$$g \circ f((x_1, x_2)) = (0, 1) \text{ for all } (x_1, x_2) \in X,$$

and

$$f \circ g((y_1, y_2)) = (0, -1) \text{ for all } (y_1, y_2) \in Y.$$

Thus three conditions of the Theorem 2.1 are satisfied.
Also as $u = (0, 1) \in X$ and $v = (0, -1) \in Y$ are the only points satisfying

$$d(u, v) \leq d(x, y) \text{ for all } x \in X \text{ and for all } y \in Y,$$

and $(0, 1) = g(0, -1)$. Thus the condition of the Corollary 2.3 is verified. In this case mapping f indeed has unique best proximity point $(0, 1)$.

The following fixed point theorem that provides both necessary and sufficient conditions for the existence of fixed points, is presented here as a corollary to Theorem 2.1.

Corollary 2.5. *Let (X, d) be a complete metric space. Let $f : X \to X$ be a continuous mapping. Then f has a fixed point if and only if there exists a continuous mapping $g : X \to X$ which satisfies the following conditions:*

(i) $g(f(x)) = f(g(f(x)))$ *for all* $x \in X$.
(ii) *For all* $x, y \in X$, $d(g(f(x)), g(f(y))) \leq \alpha d(x, y)$.
(iii) *For all* $x, y \in X$, $d(f(g(x)), f(g(y))) \leq \alpha d(x, y)$.
where $\alpha \in (0, 1)$. *Then mapping* f *has a fixed point.*

To illustrate this result, consider the following example:

Example 2.6. Consider the Euclidean two space $\mathbb{R}^2$ with the metric d, defined as

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| \text{ for all } (x_1, x_2) \text{ and } (y_1, y_2) \text{ in } \mathbb{R}^2.$$

Let $X = \{(0, 1), (0, -1), (1, 0), (-1, 0)\}$. Define $f : X \to X$ and $g : X \to X$ as

$$f((0, 1)) = (0, 1), f((0, -1)) = (-1, 0), f((1, 0)) = (1, 0), f((-1, 0)) = (0, -1),$$

and

$$g((x_1, x_2)) = (0, 1) \text{ for all } (x_1, x_2) \in X.$$

Then f and g both are continuous functions on X, satisfying all the condition of Corollary 2.5. Also mapping f have fixed points at $(0, 1)$ and $(1, 0)$.

3. An application to analytic functions of a complex variable

Theorem 3.1. *Let X and Y be two compact convex subsets of a domain D in the complex plane. Let f be an analytic function on D such that $|f'(s)| < 1$ for all $s \in X$ and $f(X) \subset Y$. Then there is a $z \in X$ such that $|z - f(z)| = d(X, Y)$ if and only if there exists an analytic function g on D such that*

(i) $g(Y) \subset X$.
(ii) $|g(f(s)) - f(gf(s))| = d(X, Y)$ *for all* $s \in X$.
(iii) $|g'(s)| < 1$ *for all* $s \in Y$.

Proof. (Necessity): Suppose there is a $z \in X$ such that $|z - f(z)| = d(X, Y)$. Define g on D as $g(s) = z$ for every $s \in D$. Then $g(Y) \subset X$ and for all $s \in X$,

$$|g(f(s)) - f(g(f(s)))| = |z - f(z)| = d(X, Y).$$

Condition (iii) is obvious.
(Sufficiency): Suppose there exists an analytic function g defined on D satisfying (i),(ii),(iii).
As $|f'(s)|$ is a continuous function on a compact set X, it attains maximum at some point, say $z \in X$.
Let $k_1 = |f'(z)|$.
Then, $k_1 < 1$ and $|f'(s)| \leq k_1$ for all $s \in X$.
Also, as $|g'(s)|$ is a continuous function on the compact set Y, it attains maximum at some point, say $z' \in Y$.
Let $k_2 = |g'(z')|$.
Then, $k_2 < 1$ and $|g'(s)| \leq k_2$ for all $s \in Y$.

Now for all s_1, s_2 in X. Let γ be the line segment connecting s_1 to s_2 in $\mathbb{C}$. Then,

$$\begin{aligned} |g(f(s_1)) - g(f(s_2))| &\leq \sup_{\xi\in\gamma} |(g \circ f)'(\xi)| \cdot |s_1 - s_2| \\ &= \sup_{\xi\in\gamma} |g'(f(\xi))f'(\xi)| \cdot |s_1 - s_2| \\ &\leq k_2k_1 \cdot |s_1 - s_2| \\ &< \alpha \cdot |s_1 - s_2|. \end{aligned}$$

where α is a real number satisfying $k_2k_1 < \alpha < 1$. An α satisfying this condition always exists when $k_2k_1 < 1$.

Similarly for all ω_1, ω_2 in Y. Let γ' be the line segment connecting ω_1 to ω_2. Then

$$\begin{aligned} |f(g(\omega_1)) - f(g(\omega_2))| &\leq \sup_{\xi\in\gamma'} |(f \circ g)'(\xi)| \cdot |\omega_1 - \omega_2| \\ &= \sup_{\xi\in\gamma'} |f'(g(\xi))g'(\xi)| \cdot |\omega_1 - \omega_2| \\ &\leq k_1k_2 \cdot |\omega_1 - \omega_2| \\ &< \alpha \cdot |\omega_1 - \omega_2|. \end{aligned}$$

Since g satisfies (ii), all the assumptions of Theorem 2.1 hold, and we obtain the required $z \in X$.

□

Corollary 3.2. *Further if $z_1 \in X$ and $z_2 \in Y$ such that $|z_1 - z_2| \leq |s_1 - s_2|$ for all $s_1 \in X$ and for all $s_2 \in Y$ implies $z_1 = g(z_2)$. Then there exists unique $z \in X$ such that*

$$|z - f(z)| = d(X, Y).$$

4. Conclusion

In this paper, we have provided a best proximity point theorem with necessary and sufficient conditions, and thus generalizing existing results and bridging a critical gap in the literature. The illustrative examples demonstrate the applicability and relevance of the theorem, including conditions for ensuring the uniqueness of best proximity points. Furthermore, a fixed point theorem was formulated as a corollary in order to characterize the relation between fixed and best proximity points. The application of these results to analytic functions of a complex variable underscores their potential in solving complex problems across diverse mathematical and applied domains. Future researcher may focus on further exploring applications in other fields and extending the results to broader classes of spaces and mappings.

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References

[1] M. Al-Thagafi, and N. Shahzad, Convergence and existence results for best proximity points, Nonlinear Anal. 70, no. 10 (2009), 3665–3671.

[2] M. Aslantaş, Best proximity point theorems for proximal b-cyclic contractions on b-metric spaces, Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat. 70, no. 1 (2021), 483–496.

[3] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3, no. 1 (1922), 133–181.

[4] S. S. Basha, Extensions of Banach's contraction principle, Numer. Funct. Anal. Optim. 31, no. 5 (2010), 569–576.

[5] S. S. Basha, Best proximity points: global optimal approximate solutions, J. Global Optim. 49, no. 1 (2011), 15–21.

[6] S. S. Basha, Some best approximation theorems and best proximity point theorems, Acta Sci. Math. (Szeged) (2023), 1–12.

[7] S. S. Basha, and N. Shahzad, Best proximity point theorems for generalized proximal contractions, Fixed Point Theory Appl. 2012 (2012), 1–9.

[8] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum 9 (2004), 43–54.

[9] D. W. Boyd, and J. S. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20, no. 2 (1969), 458–464.

[10] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215 (1976), 241–251.

[11] J. Caristi, and W. A. Kirk, Geometric fixed point theory and inwardness conditions, in: The Geometry of Metric and Linear Spaces, Proc. Conf., Michigan State Univ., East Lansing, June 17–19, 1974, (2006), 74–83.

[12] N. Chandra, B. Joshi, and M. C. Joshi, Generalized fixed point theorems on metric spaces, Math. Morav. 26, no. 2 (2022), 85–101.

[13] S. K. Chatterjea, Fixed-point theorems, Dokl. Bulg. Akad. Nauk 25, no. 6 (1972), 727–730.

[14] L. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45, no. 2 (1974), 267–273.

[15] C. Di Bari, T. Suzuki, and C. Vetro, Best proximity points for cyclic Meir–Keeler contractions, Nonlinear Anal. 69, no. 11 (2008), 3790–3794.

[16] M. Edelstein, An extension of Banach's contraction principle, Proc. Amer. Math. Soc. 12, no. 1 (1961), 7–10.

[17] A. A. Eldred, and P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323, no. 2 (2006), 1001–1006.

[18] M. Gabeleh, Best proximity point theorems via proximal non-self mappings, J. Optim. Theory Appl. 164, no. 2 (2015), 565–576.

[19] D. Gopal, P. Kumam, and M. Abbas, Background and recent developments of metric fixed point theory, Chapm. & Hall/CRC (2017).

[20] M. Jleli, and B. Samet, A new generalization of the Banach contraction principle, J. Inequal. Appl. 2014 (2014), 1–8.

[21] R. Kannan, Some results on fixed points–II, Amer. Math. Monthly 76, no. 4 (1969), 405–408.

[22] S. Karpagam, and S. Agrawal, Best proximity point theorems for p-cyclic Meir–Keeler contractions, Fixed Point Theory Appl. 2009 (2009), 1–9.

[23] E. Keeler, and A. Meir, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326–329.

[24] F. Khojasteh, S. Shukla, and S. Radenović, A new approach to the study of fixed point theory for simulation functions, Filomat 29, no. 6 (2015), 1189–1194.

[25] W. A. Kirk, and B. Sims, Handbook of metric fixed point theory, Springer Dordrecht (2002).

[26] E. Rakotch, A note on contractive mappings, Proc. Amer. Math. Soc. 13, no. 3 (1962), 459–465.

[27] B. E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257–290.

[28] D. Sarkhel, Banach's fixed point theorem implies Kannan's, Bull. Cal. Math. Soc. 91, no. 2 (1999), 143–144.

[29] N. K. Singh, and S. Pant, Generalization of Kannan results for metric interval space without space completeness and application to obtain near fixed interval ellipse, Appl. Math. E-Notes 25 (2025), 15– 30.

[30] P. Subrahmanyam, Remarks on some fixed point theorems related to Banach's contraction principle, J. Math. Phys. Sci. 8 (1974), 445–457.

[31] T. Suzuki, Generalized distance and existence theorems in complete metric spaces, J. Math. Anal. Appl. 253, no. 2 (2001), 440–458.

[32] T. Suzuki, Several fixed point theorems concerning φ-distance, Fixed Point Theory Appl., 2004 (2004), 1–15.

[33] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136, no. 5 (2008), 1861–1869.

[34] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), 1–6.

[35] M. Yadav, B. Thakur, and A. Sharma, Best proximity points for generalized proximity contraction in complete metric spaces, Adv. Fixed Point Theory 3, no. 2 (2013), 392–405.